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ON THE NUMBER OF POLES OF THE DYNAMICAL ZETA FUNCTIONS FOR BILLIARD FLOW

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Abstract. We study the number of the poles of the meromorphic continuation of the dynamical zeta functions η_N and η_D for several strictly convex disjoint obstacles satisfying the non-eclipse condition. For η_N, we obtain a strip $\{z \in \mathbb{C} : \mathrm{Re}\, s > \beta\}$ with infinitely many poles. For η_D, we prove the same result assuming the boundary is real analytic. Moreover, for η_N we obtain a characterization of β by the pressure $\mathrm{P}(2G)$ of some function G on the space Σ_A^f related to the dynamical characteristics of the obstacle.

1. **Introduction.** Let $D_1, \dots, D_r \subset \mathbb{R}^d$, $r \geqslant 3$, $d \geqslant 2$, be compact strictly convex disjoint obstacles with C^∞ smooth boundary, and let $D = \bigcup_{j=1}^r D_j$. We assume that every D_j has non-empty interior and throughout this paper we suppose the following non-eclipse condition:

$$D_k \cap \text{convex hull}\,(D_i \cup D_j) = \emptyset, \tag{1}$$

for any $1 \leqslant i, j, k \leqslant r$ such that $i \neq k$ and $j \neq k$. Under this condition, all periodic trajectories for the billiard flow in $\Omega = \mathbb{R}^d \setminus \mathring{D}$ are ordinary reflecting ones without tangent segments to the boundary of D. We consider the (non-grazing) billiard flow φ_t (see Section 2 for the definition). Next, the periodic trajectories will be called periodic rays. For any periodic ray γ, denote by $\tau(\gamma) > 0$ its period, by $\tau^\sharp(\gamma) > 0$ its primitive period, and by $m(\gamma)$ the number of reflections of γ at the obstacles. Denote by P_γ the associated linearized Poincaré map (see section 2.3 in [24] and Section 2 for the definition). Let $\mathcal{P}$ be the set of all oriented periodic rays. Notice that some periodic rays have only one orientation, while others admits two (see [4, §2.3] for more details). Let Π be the set of all primitive periodic rays. Then, the counting function of the lengths of periodic rays satisfies

$$\sharp\{\gamma \in \Pi : \tau^\sharp(\gamma) \leq x\} \sim \frac{\mathrm{e}^{hx}}{hx}, \quad x \to +\infty, \tag{2}$$

for some $h > 0$ (see for instance [20, Theorem 6.5] for weakly mixing suspension symbolic flow, and [15] and [19]). Hence, there exists an infinite number of primitive

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periodic trajectories and applying (2), for every sufficiently small $\epsilon > 0$, one obtains the estimate

$$e^{(h-\epsilon)x} \leq \sharp\{\gamma \in \mathcal{P} : \tau(\gamma) \leq x\} \leq \mathrm{e}^{(h+\epsilon)x},\ x \geq C_\epsilon \gg 1.$$

Moreover, for some positive constants c_1, C_1, f_1 and f_2, we have (see for instance [21, Appendix] and (A.1))

$$c_1 e^{f_1 \tau(\gamma)} \leq |\det(\mathrm{Id} - P_\gamma)| \leq C_1 e^{f_2 \tau(\gamma)},\ \gamma \in \mathcal{P}.$$

By using these estimates, define for $\mathrm{Re}\,(s) \gg 1$ two Dirichlet series

$$\eta_{\mathrm{N}}(s) = \sum_{\gamma \in \mathcal{P}} \frac{\tau^\sharp(\gamma)\mathrm{e}^{-s\tau(\gamma)}}{|\det(\mathrm{Id} - P_\gamma)|^{1/2}}, \quad \eta_{\mathrm{D}}(s) = \sum_{\gamma \in \mathcal{P}} (-1)^{m(\gamma)} \frac{\tau^\sharp(\gamma)\mathrm{e}^{-s\tau(\gamma)}}{|\det(\mathrm{Id} - P_\gamma)|^{1/2}},$$

where the sums run over all oriented periodic rays. The length $\tau^\sharp(\gamma)$, the period $\tau(\gamma)$ and $|\det(\mathrm{Id} - P_\gamma)|^{1/2}$ are independent of the orientation of γ. We also consider for $q \geq 1,\ q \in \mathbb{N}$, the zeta function

$$\eta_{\mathrm{q}}(s) = q \sum_{\gamma \in \mathcal{P}, m(\gamma) \in q\mathbb{N}} \frac{\tau^\sharp(\gamma)\mathrm{e}^{-s\tau(\gamma)}}{|\det(\mathrm{Id} - P_\gamma)|^{1/2}}, \mathrm{Re}\, s \gg 1.$$

Clearly, $\eta_N(s) = \eta_1(s)$. These zeta functions are important for the analysis of the distribution of the scattering resonances related to the Laplacian in $\mathbb{R}^d \setminus \bar{D}$ with Dirichlet and Neumann boundary conditions on ∂D (see [4, §1] for more details).

It was proved in [4, Theorem 1.1 and Theorem 4.1] that η_q has a meromorphic continuation to $\mathbb{C}$ with simple poles and integer residues. We have the equality

$$\eta_D(s) = \eta_2(s) - \eta_1(s),\ \mathrm{Re}\, s \gg 1, \tag{3}$$

hence η_D admits also a meromorphic continuation to $\mathbb{C}$ with simple poles and integer residues. The functions $\eta_q(s)$ are Dirichlet series with positive coefficients and by a classical theorem of Landau (see for instance, [2, Théorème 1, Chapitre IV]), they have a pole $s = a_q$, where a_q is the abscissa of convergence of $\eta_q(s)$. On the other hand, from (3) it follows that some cancellations of poles are possible. In this direction, for $d = 2$ [27] and for $d \geqslant 3$ under some conditions [29], Stoyanov proved that there exists $\varepsilon > 0$ such that $\eta_{\mathrm{D}}(s)$ is analytic for $\mathrm{Re}\, s \geqslant a_1 - \varepsilon$. The same result has been proved for $d = 3$ and $a_1 > 0$ by Ikawa [16].

The purpose of this paper is to prove that $\eta_q(s)$ has an infinite number of poles and to estimate $\beta \in \mathbb{R}$ such that the number of poles with $\mathrm{Re}\, s > \beta$ is infinite. The same questions are more difficult for $\eta_D(s)$ since the existence of at least one pole has been established only for obstacles with real analytic boundary [4, Theorem 1.3], and for obstacles with sufficiently small diameters [15], [28]. Clearly, $a_q \leq a_1$. We have $a_2 = a_1$, since if $a_2 < a_1$, the function η_D will have a singularity at a_1 which is impossible because η_D is analytic for $\mathrm{Re}\, s \geq a_1$ (see [21, Theorem 1]).

Denote by $\mathrm{Res}\, \eta_q$ and $\mathrm{Res}\, \eta_D$ the set of poles of η_q and η_D, respectively. We prove the following theorem.

Theorem 1.1. *For every $0 < \delta < 1$, there exists $\alpha_{\delta,q} < a_q$ such that for $\alpha < \alpha_{\delta,q}$, we have*

$$\sharp\{\mu_j \in \mathrm{Res}\, \eta_q : \mathrm{Re}\, \mu_j \geq \alpha,\ |\mu_j| \leqslant r\} \neq \mathcal{O}(r^\delta). \tag{4}$$

If η_D cannot be prolonged as an entire function, the same result holds for $\mathrm{Res}\, \eta_D$.

More precisely, we show that for any $0 < \delta < 1$ there exists $\alpha_{\delta,q} < 0$ depending on the dynamical characteristics of D such that if $\alpha < \alpha_{\delta,q}$, for any constant $0 < C < \infty$ the estimate

$$\sharp\{\mu_j \in \mathrm{Res}\, \eta_q :\ \mathrm{Re}\, \mu_j \geq \alpha,\ |\mu_j| \leqslant r\} \leqslant C r^{\delta}, \quad r \geqslant 1$$

does not hold. Similar results have been proved for Pollicott-Ruelle resonances for Anosov flows [18, Theorem 2] and for Axiom A flows [17, Theorem 4.1]. For obstacles D satisfying (1) the same result for scattering resonances has been proved for Neumann problem in [22] and for Dirichlet problem and real analytic boundary in [4, Theorem 1.3]. According to Theorem 1.1, it follows that for large $A > 0$ in the region $\mathcal{D}_A = \{z \in \mathbb{C} : \mathrm{Re}\, z > -A\}$ there are infinitely many poles $\mu \in \mathrm{Res}\, \eta_1 \cap \mathcal{D}_A$ and infinitely many poles $\nu \in \mathrm{Res}\, \eta_2 \cap \mathcal{D}_A$. Therefore, if η_D is analytic in $\mathcal{D}_A$, by (3) we deduce that we must have an *infinite number of cancellations* of poles μ with poles ν and the corresponding residues of the cancelled poles μ and ν must coincide.

Remark 1.2. The proof of Theorem 1.1 for η_D works under the condition that there exist two sequences $(\ell_j), (m_j)$ with $\ell_j \to \infty$, $m_j \to \infty$ as $j \to \infty$ with the properties in Proposition 4.1. If η_D cannot be prolonged as an entire function, the existence of such sequences has been established by Ikawa (see [14, Prop. 2.3]). In Theorem 1.1 we prove the inverse result.

It is interesting to find the supremum of numbers $\beta_q < a_q$ such that the strip $\{z \in \mathbb{C} :\ \mathrm{Re}\, z > \beta_q\}$ contains an infinite number poles of η_q and to obtain so called *essential spectral gap*. This is a difficult open problem. Let $b_q < a_q$ be the abscissa of convergence of the series

$$\sum_{\gamma, m(\gamma) \in q\mathbb{N}} \frac{\tau^{\sharp}(\gamma) e^{-s\tau(\gamma)}}{|\det(\mathrm{Id} - P_\gamma)|},\ \mathrm{Re}\, s \gg 1 \tag{5}$$

and let $\alpha = \max\{0, a_1\}$. In our second result, we obtain a more precise result for $\mathrm{Res}\, \eta_1$.

Theorem 1.3. *For any small $\epsilon > 0$, we have*

$$\sharp\{\mu_j \in \mathrm{Res}\, \eta_1 :\ \mathrm{Re}\, \mu_j > (2d^2 + 2d - 1/2)(b_1 - 2\alpha) - \epsilon\} = \infty. \tag{6}$$

Notice that

$$2d^2 + 2d - 1/2 = 2(d^2 + d - 1) + 3/2 = 2 \dim G + 3/2,$$

where G is the $(d-1)$-Grassmannian bundle introduced in Section 2. In the appendix, we prove that b_1 coincides with the abscissa of convergence of the series

$$\sum_{\gamma} \frac{\tau^{\sharp}(\gamma) e^{-s\tau(\gamma)}}{|\det(D_x \varphi_{\tau(\gamma)}|_{E_u(x)})|},\ \mathrm{Re}\, s \gg 1, \tag{7}$$

where $E_u(x)$ is the unstable space of $x \in \gamma$ (see (10) for the notation). By using symbolic dynamics, we define in (A.3) a function $G(\xi, y) < 0$ on the space Σ_A^f related to the dynamical characteristics of D (see the appendix for definitions) and prove the following

Proposition 1.4. *The abscissas of convergence a_1 and b_1 are given by*

$$a_1 = \mathrm{P}(G),\ b_1 = \mathrm{P}(2G), \tag{8}$$

$\mathrm{P}(G)$ *being the pressure of G defined by (A.2).*

For $a_1 \leq 0$, we have $\alpha = 0$, and Theorem 1.3 is similar to [18, Theorem 3] established for weakly mixing Anosov flows ψ_t, where instead of $b_1 = \mathrm{P}(2G)$, one has the pressure $\mathrm{P}(2\psi^u) < 0$ of the Sinai-Ruelle-Bowen potential

$$\psi^u(x) = -\frac{d}{dt}\Big(\log|\mathrm{det} D_x\psi_t|_{E_u(x)}|\Big)|_{t=0}.$$

Notice that for Anosov flow one has $\mathrm{P}(\psi^u) = 0$ (see [3, Theorem 5]), while in our case $a_1 = \mathrm{P}(G)$ can be different from 0. More precise results for the poles of the semi-classical zeta function for contact Anosov flows have been obtained in [8], [9, Theorem 1.2].

Remark 1.5. The constant $2d^2 + 2d - 1/2$ in (6) is related to the estimate (22) of the Fourier transform $\hat{F}_{A,1}$ in the local trace formula for $\eta_1(s)$ in Theorem 3.2 and is probably not optimal. A better estimate of $\hat{F}_{A,1}$ can be obtained if the bound of the number of poles (22) is improved (see for example, [1], where the Hausdorff dimension of the trapped set K is involved).

We have $b_1 = b_2$ since the series

$$\sum_{\gamma} \frac{(-1)^{m(\gamma)}\tau^\sharp(\gamma)e^{-s\tau(\gamma)}}{|\det(\mathrm{Id} - P_\gamma)|},\ \mathrm{Re}\, s \gg 1 \tag{9}$$

is analytic for $\mathrm{Re}\, s \geq b_1$. We discuss this question at the end of the appendix. Theorem 1.3 can be generalized for $\mathrm{Res}\, \eta_2$ and one obtains (6). The proof works with some modifications.

The paper is organized as follows. In Section 2 we collect some definitions and notations from [4] which are necessary for the exposition. In particular, we define the non-grazing billiard flow φ_t, the $(d-1)$-Grassmannian bundle G, the bundles $\mathcal{E}_{k,\ell}$ over G, and the operators $\mathbf{P}_{k,\ell}$, $0 \leq k \leq d$, $0 \leq \ell \leq d^2$. In Section 3 we obtain local trace formulas combining the results in [17, §6.1] and [4, Lemma 3.1]. In Section 4 we prove Theorems 1.1 and 1.3. Finally, in the appendix we use symbolic dynamics and establish Proposition 1.4.

2. **Preliminaries.** We recall the definition of billiard flow ϕ_t described in [4, §2.1]. Denote by $S\mathbb{R}^d$ the unit tangent bundle of $\mathbb{R}^d$, and by $\pi : S\mathbb{R}^d \to \mathbb{R}^d$ the natural projection. For $x \in \partial D_j$, denote by $n_j(x)$ the *inward unit normal vector* to ∂D_j at the point x pointing into D_j. Set

$$\mathcal{D} = \{(x, v) \in S\mathbb{R}^d \ : \ x \in \partial D\}.$$

We say that $(x, v) \in T_{\partial D_j}(\mathbb{R}^d)$ is incoming (resp. outgoing) if we have $\langle v, n_j(x)\rangle > 0$ (resp. $\langle v, n_j(x)\rangle < 0$). Introduce

$$\mathcal{D}_{\mathrm{in}} = \{(x, v) \in \mathcal{D} \ : \ (x, v) \text{ is incoming}\},$$
$$\mathcal{D}_{\mathrm{out}} = \{(x, v) \in \mathcal{D} \ : \ (x, v) \text{ is outgoing}\}.$$

Define the grazing set $\mathcal{D}_{\mathrm{g}} = T(\partial D) \cap \mathcal{D}$ and obtain

$$\mathcal{D} = \mathcal{D}_{\mathrm{g}} \sqcup \mathcal{D}_{\mathrm{in}} \sqcup \mathcal{D}_{\mathrm{out}}.$$

The billiard flow $(\phi_t)_{t\in\mathbb{R}}$ is the complete flow acting on $S\mathbb{R}^d \setminus \pi^{-1}(\mathring{D})$, which is defined as follows. For $(x, v) \in S\mathbb{R}^d \setminus \pi^{-1}(\mathring{D})$, set

$$\tau_\pm(x, v) = \pm \inf\{t \geqslant 0 : x \pm tv \in \partial D\}.$$

For $(x, v) \in \mathcal{D}_{\mathrm{in/out}}$, denote by $v' \in \mathcal{D}_{\mathrm{out/in}}$ the image of v by the reflexion with respect to $T_x(\partial D)$ at $x \in \partial D_j$, given by

$$v' = v - 2\langle v, n_j(x)\rangle n_j(x), \quad v \in S_x(\mathbb{R}^d), \quad x \in \partial D_j.$$

Then, for $(x, v) \in (S\mathbb{R}^d \setminus \pi^{-1}(D)) \cup \mathcal{D}_{\mathrm{g}}$, define

$$\phi_t(x, v) = (x + tv, v), \quad t \in [\tau_-(x, v), \tau_+(x, v)],$$

while for $(x, v) \in \mathcal{D}_{\mathrm{in/out}}$, we set

$$\phi_t(x, v) = (x + tv, v) \quad \text{if} \quad \begin{cases} (x, v) \in \mathcal{D}_{\mathrm{out}},\ t \in [0, \tau_+(x, v)]\,, \\ \text{or } (x, v) \in \mathcal{D}_{\mathrm{in}},\ t \in [\tau_-(x, v), 0]\,, \end{cases}$$

and

$$\phi_t(x, v) = (x + tv', v') \quad \text{if} \quad \begin{cases} (x, v) \in \mathcal{D}_{\mathrm{in}},\ t \in \,]0, \tau_+(x, v)]\,, \\ \text{or } (x, v) \in \mathcal{D}_{\mathrm{out}},\ t \in [\tau_-(x, v'), 0[\,. \end{cases}$$

We extend ϕ_t to a complete flow still denoted by ϕ_t, having the property

$$\phi_{t+s}(x, v) = \phi_t(\phi_s(x, v)), \quad t, s \in \mathbb{R}, \quad (x, v) \in S\mathbb{R}^d \setminus \pi^{-1}(\mathring{D}).$$

Next, we introduce the non-grazing set M as

$$M = B/\sim, \quad B = S\mathbb{R}^d \setminus \left(\pi^{-1}(\mathring{D}) \cup \mathcal{D}_{\mathrm{g}}\right),$$

where $(x, v) \sim (y, w)$ if and only if $(x, v) = (y, w)$ or

$$x = y \in \partial D \quad \text{and} \quad w = v'.$$

The set M is endowed with the quotient topology. We change the notation and pass from ϕ_t to the non-grazing flow φ_t, which is defined on M as follows. For $(x, v) \in (S\mathbb{R}^d \setminus \pi^{-1}(D)) \cup \mathcal{D}_{\mathrm{in}}$, define

$$\varphi_t([(x, v)]) = [\phi_t(x, v)], \quad t \in \left]\tau_-^{\mathrm{g}}(x, v), \tau_+^{\mathrm{g}}(x, v)\right[,$$

where $[z]$ denotes the equivalence class of $z \in B$ for the relation $\sim$, and

$$\tau_\pm^{\mathrm{g}}(x, v) = \pm \inf\{t > 0 : \phi_{\pm t}(x, v) \in \mathcal{D}_{\mathrm{g}}\}.$$

Thus, φ_t is continuous, but the flow trajectory of (x, v) for times $t \notin \left]\tau_-^{\mathrm{g}}(x, v), \tau_+^{\mathrm{g}}(x, v)\right[$ is not defined. Clearly, we may have $\tau_\pm^{\mathrm{g}}(x, v) = \pm\infty$, while $\tau_\pm^{\mathrm{g}}(x, v) \neq 0$ for $(x, v) \in \mathcal{D}_{\mathrm{in}}$. Note that the above formula indeed defines a flow on M because each $(x, v) \in B$ has a unique representative in $(S\mathbb{R}^d \setminus \pi^{-1}(\mathring{D})) \cup \mathcal{D}_{\mathrm{in}}$. Following [6, Theorem 3.2], we may define smooth charts on $M = B/\sim$ and φ_t becomes C^∞ a non-complete flow with respect to new charts.

Throughout, we work with the smooth flow φ_t and denote by X its generator. Let $A(z) = \{t \in \mathbb{R} : \pi(\varphi_t(z)) \in \partial D\}$. The *trapped set* K of φ_t is the set of points $z \in M$ which satisfy $-\tau_-^{\mathrm{g}}(z) = \tau_+^{\mathrm{g}}(z) = +\infty$ and

$$\sup A(z) = -\inf A(z) = +\infty.$$

By definition, $\varphi_t(z)$ is defined for all $t \in \mathbb{R}$ whenever $z \in K$. The flow φ_t is called *uniformly hyperbolic* on K if for each $z \in K$ there exists a $\mathrm{d}\varphi_t$ invariant decomposition

$$T_z M = \mathbb{R}X(z) \oplus E_u(z) \oplus E_s(z), \tag{10}$$

with $\dim E_s(z) = \dim E_u(z) = d-1$, such that for some constants $C > 0$, $\nu > 0$ independent of $z \in K$ and some smooth norm $\|\cdot\|$ on TM, we have

$$\|\mathrm{d}\varphi_t(z)\cdot v\| \leqslant \begin{cases} C\mathrm{e}^{-\nu t}\|v\|, & v \in E_s(z), \quad t \geqslant 0, \\ C\mathrm{e}^{-\nu|t|}\|v\|, & v \in E_u(z), \quad t \leqslant 0. \end{cases} \tag{11}$$

The spaces $E_s(z)$ and $E_u(z)$ depend continuously on z (see [11, Section 2]).

The flow φ_t is uniformly hyperbolic on K (for the proof see [4, Appendix A]). Take a small neighborhood V of K in M with smooth boundary, and embed V into a compact manifold without boundary N. We arbitrarily extend X to obtain a smooth vector field on N, still denoted by X. The associated flow is still denoted by φ_t. Note that the new flow φ_t is now complete. Introducing the surjective map

$$\pi_M : B \ni (x, \xi) \to [(x, \xi)] \in M,$$

we have $\varphi_t \circ \pi_M = \pi_M \circ \phi_t$. There is a bijection between periodic orbits of ϕ_t and φ_t preserving the periods of the closed trajectories of ϕ_t, while the corresponding Poincaré maps are conjugated (see [6, Section 3]).

Consider the $(d-1)$-Grassmannian bundle

$$\pi_G : G \to N$$

over N. More precisely, for every $z \in N$, the set $\pi_G^{-1}(z)$ consists of all $(d-1)$-dimensional planes of T_zN. The dimension of $\pi_G^{-1}(z)$ is $d(d-1)$ and G is a smooth compact manifold with $\dim G = d^2 + d - 1$. We lift φ_t to a flow $\widetilde{\varphi}_t : G \to G$ defined by

$$\widetilde{\varphi}_t(z, E) = (\varphi_t(z), \mathrm{d}\varphi_t(z)(E)),\ z \in N,\ E \subset T_zN,\ \mathrm{d}\varphi_t(z)(E) \subset T_{\varphi_t(z)}N.$$

Introduce the set

$$\widetilde{K}_u = \{(z, E_u(z)) \ : \ z \in K\} \subset G.$$

Clearly, $\widetilde{K}_u$ is invariant under the action of $\widetilde{\varphi}_t$, since $\mathrm{d}\varphi_t(z)(E_u(z)) = E_u(\varphi_t(z))$. The set $\widetilde{K}_u$ will be seen as the trapped set of the restriction of $\widetilde{\varphi}_t$ to a neighborhood of $\widetilde{K}_u$ and the flow $\widetilde{\varphi}_t$ is uniformly hyperbolic on $\widetilde{K}$(see [3, Lemma A.3] and [4, §2.5]). Let $\widetilde{X}$ be the generator of $\widetilde{\varphi}_t$ and let $\widetilde{V}_u$ be a small neighborhood of $\widetilde{K}_u$ in G with smooth boundary $\partial\widetilde{V}_u$ (see [4, §2.7]). Define

$$\Gamma_\pm(\widetilde{X}) = \{z \in \widetilde{V}_u : \ \widetilde{\varphi}_t(z) \in \widetilde{V}_u,\ \mp t > 0\}.$$

Denote by clos $\widetilde{V}_u$ the closure of $\widetilde{V}_u$. Let $\tilde{\rho} \in C^\infty(\text{clos } \widetilde{V}_u, \bar{\mathbb{R}}_+)$ be the defining function for $\widetilde{V}_u$ such that $\partial\widetilde{V}_u = \{z \in \text{clos } \widetilde{V}_u : \tilde{\rho}(z) = 0\}$ and $\mathrm{d}\tilde{\rho}(z) \neq 0$ for any $z \in \partial\widetilde{V}_u$. Following [10, Lemma 2.3], for any small neighborhood $\widetilde{W}_0$ of $\partial\widetilde{V}_u$, there exists a vector field $\widetilde{Y}$ on clos $\widetilde{V}_u$ arbitrary close to $\widetilde{X}$ in C^∞-topology and flow $\widetilde{\psi}_t$ generated by $\widetilde{Y}$ with the properties:

$$(1)\ \text{supp}\,(\widetilde{Y} - \widetilde{X}) \subset \widetilde{W}_0.$$

(2) (Convexity condition) For any defining function ρ of $\widetilde{V}_u$ and any $\omega \in \partial\widetilde{V}_u$, we have

$$\widetilde{Y}\rho(\omega) = 0 \Longrightarrow \widetilde{Y}^2\rho(\omega) < 0.$$

(3) $\Gamma_\pm(\widetilde{X}) = \Gamma_\pm(\widetilde{Y})$, where $\Gamma_\pm(\widetilde{Y})$ is defined as above by $\widetilde{\psi}_t$.

By [7, Lemma 1.1], we may find a smooth extension of $\widetilde{Y}$ on G (still denoted by $\widetilde{Y}$) so that for every $\omega \in G$ and $T \geqslant 0$, we have

$$\omega, \widetilde{\psi}_T(\omega) \in \text{clos } \widetilde{V}_u \quad \Longrightarrow \quad \widetilde{\psi}_t(\omega) \in \text{clos } \widetilde{V}_u,\ \forall t \in [0, T]. \tag{12}$$

In the following, we fix $\widetilde{V}_u, \widetilde{W}_0, \tilde{Y}$ and the flow $\tilde{\psi}_t$ with the properties mentioned above. Thus, we obtain an *open hyperbolic system* satisfying conditions (**A1**)−(**A4**) in [7, §0] (see also [17, §2.1]).

Next, repeating the setup in [4, §2.6], we introduce some bundles passing to open hyperbolic system for bundles. First, define the tautological vector bundle $\mathcal{E} \to G$ by

$$\mathcal{E} = \{(\omega, u) \in \pi_G^*(TN) \ : \ \omega \in G, \ u \in [\omega]\},$$

where $[\omega] = E$ denotes the $(d-1)$ dimensional subspace of $T_{\pi_G(\omega)}N$ represented by $\omega = (z, E)$, and $\pi_G^*(TN)$ is the pullback bundle of TN. Second, introduce the "vertical bundle" $\mathcal{F} \to G$ by

$$\mathcal{F} = \{(\omega, W) \in TG \ : \ \mathrm{d}\pi_G(\omega) \cdot W = 0\},$$

which is a subbundle of the bundle $TG \to G$. The dimensions of the fibres $\mathcal{E}_\omega$ and $\mathcal{F}_\omega$ of $\mathcal{E}$ and $\mathcal{F}$ over ω are given by

$$\dim \mathcal{E}_\omega = d - 1, \quad \dim \mathcal{F}_\omega = \dim \ker \mathrm{d}\pi_G(\omega) = d^2 - d$$

for any $\omega \in G$ with $\pi_G(\omega) = z$. Finally, set

$$\mathcal{E}_{k,\ell} = \wedge^k \mathcal{E}^* \otimes \wedge^\ell \mathcal{F}, \quad 0 \leqslant k \leqslant d-1, \quad 0 \leqslant \ell \leqslant d^2 - d,$$

where $\mathcal{E}^*$ is the dual bundle of $\mathcal{E}$, that is, we replace the fibre $\mathcal{E}_\omega$ by its dual space $\mathcal{E}_\omega^*$.

Next, we use the notation $\omega = (z, \eta) \in G$ and $u \otimes v \in \mathcal{E}_{k,\ell}|_\omega$. By using the flow $\tilde{\psi}_t$, introduce a flow $\Phi_t^{k,\ell} : \mathcal{E}_{k,\ell} \to \mathcal{E}_{k,\ell}$ by

$$\Phi_t^{k,\ell}(\omega, u \otimes v) = \Big(\tilde{\psi}_t(\omega), \ b_t(\omega) \cdot \Big[\big(\mathrm{d}\varphi_t(\pi_G(\omega))^{-\top}\big)^{\wedge k}(u) \otimes \mathrm{d}\widetilde{\psi}_t(\omega)^{\wedge \ell}(v)\Big]\Big), \quad (13)$$

with

$$b_t(\omega) = |\det \mathrm{d}\varphi_t(\pi_G(\omega))|_{[\omega]}|^{1/2} \cdot |\det \Big(\mathrm{d}\tilde{\psi}_t(\omega)|_{\ker \mathrm{d}\pi_G}\Big)|^{-1},$$

where $^{-\top}$ denotes the inverse transpose. Consider the transfer operator

$$\Phi_{-t}^{k,\ell,*} : C^\infty(G, \mathcal{E}_{k,\ell}) \to C^\infty(G, \mathcal{E}_{k,\ell})$$

defined by

$$\Phi_{-t}^{k,\ell,*}\mathbf{u}(\omega) = \Phi_t^{k,\ell}\big[\mathbf{u}(\widetilde{\psi}_{-t}(\omega))\big], \quad \mathbf{u} \in C^\infty(G, \mathcal{E}_{k,\ell}) \qquad (14)$$

and let $\mathbf{P}_{k,\ell} : C^\infty(G, \mathcal{E}_{k,\ell}) \to C^\infty(G, \mathcal{E}_{k,\ell})$ be the generator of $\Phi_{-t}^{k,\ell,*}$ given by

$$\mathbf{P}_{k,\ell}\mathbf{u} = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\Phi_{-t}^{k,\ell,*}\mathbf{u}\Big)\Big|_{t=0}, \quad \mathbf{u} \in C^\infty(G, \mathcal{E}_{k,\ell}).$$

The operator $\mathbf{P}_{k,\ell}$ has the property

$$\mathbf{P}_{k,\ell}(f\mathbf{u}) = (\mathbf{P}_{k,\ell}f)\mathbf{u} + f(\mathbf{P}_{k,\ell}\mathbf{u}), \ f \in C^\infty(G), \ \mathbf{u} \in C^\infty(G, \mathcal{E}_{k,\ell}).$$

Thus, we obtain the same setup as in Definition 6.1 in [17, §6.1]. In the last paper the authors deal with a general Axiom A flow with several basic sets. In our case, we have only one basic set and we may apply the results of [7] and [17]. With some constant $C > 0$, we have

$$\|e^{-t\mathbf{P}_{k,\ell}}\|_{L^2(G,\mathcal{E}_{k,\ell}) \to L^2(G,\mathcal{E}_{k,\ell})} \leq Ce^{Ct}, \ t \geq 0$$

and

$$(\mathbf{P}_{k,\ell} + s)^{-1} = \int_0^\infty e^{-t(\mathbf{P}_{k,\ell}+s)}dt : L^2(G, \mathcal{E}_{k,\ell}) \to L^2(G, \mathcal{E}_{k,\ell}), \ \mathrm{Re}\, s \gg 1.$$

Introduce the operator

$$\mathbf{R}_{k,\ell}(s) = \mathbf{1}_{\widetilde{V}_u}(\mathbf{P}_{k,\ell}+s)^{-1}\mathbf{1}_{\widetilde{V}_u} : C_c^\infty(\widetilde{V}_u, \mathcal{E}_{k,\ell}) \to \mathcal{D}'(\widetilde{V}_u, \mathcal{E}_{k,\ell}), \quad \mathrm{Re}\,(s) \gg 1,$$

where $\mathcal{D}'(\widetilde{V}_u, \mathcal{E}_{k,\ell})$ denotes the space of $\mathcal{E}_{k,\ell}$-valued distributions. Applying [7, Theorem 1], we obtain a meromorphic extension of $\mathbf{R}_{k,\ell}(s)$ to the whole plane $\mathbb{C}$ with simple poles and positive integer residues.

For $\omega \in G$ and $t > 0$, consider the *parallel transport* map

$$\alpha_{\omega,t}^{k,\ell} = \alpha_{1,\omega,t} \otimes \alpha_{2,\omega,t} : \Lambda^k \mathcal{E}_\omega^* \otimes \Lambda^\ell \mathcal{F}_\omega \longrightarrow \Lambda^k \mathcal{E}^*_{\tilde{\psi}_t(\omega)} \otimes \Lambda^\ell \mathcal{F}_{\tilde{\psi}_t(\omega)}$$

given by

$$\mathbf{u} \otimes \mathbf{v} \longmapsto (e^{-t\mathbf{P}_{k,\ell}}(\mathbf{u}\otimes\mathbf{v}))(\widetilde{\psi}(t)),$$

where $\mathbf{u}, \mathbf{v}$ are some sections of $\mathcal{E}_\omega^*$ and $\mathcal{F}_\omega$ over ω, respectively. The definition does not depend on the choice of $\mathbf{u}$ and $\mathbf{v}$ (see [7, Eq. (0.8)]). For a periodic trajectory $\widetilde{\gamma} : t \to \widetilde{\gamma}(t) = (\gamma(t), E_u(\gamma(t)))$ with period T, we define

$$\mathrm{tr}(\alpha_{\tilde{\gamma}}^{k,\ell}) = \mathrm{tr}(\alpha^{k,\ell}_{\tilde{\gamma}(t),T})$$

(see [7], [4]) and the trace is independent of the choice of the point $\widetilde{\gamma}(t) \in \widetilde{\gamma}$.

Finally, if $\widetilde{\chi} \in C_c^\infty(\widetilde{V}_u)$ is equal to 1 near the trapping set $\widetilde{K}_u$, we have the Guillemin trace formula (see [7, (4.6)], [25, §3.1], [4, §3.2]) with flat trace

$$\mathrm{tr}^\flat(\widetilde{\chi}e^{-t\mathbf{P}_{k,\ell}}\widetilde{\chi}) = \sum_{\widetilde{\gamma}} \frac{\tau^\sharp(\gamma)\mathrm{tr}(\alpha_{\tilde{\gamma}}^{k,\ell})\delta(t-\tau(\gamma))}{|\det(\mathrm{Id} - \widetilde{P}_\gamma)|},\ t > 0. \tag{15}$$

Here, both sides are distributions on $(0,\infty)$, the sum runs over all periodic orbits $\tilde{\gamma}$ of $\tilde{\varphi}_t$,

$$\widetilde{P}_\gamma = \mathrm{d}\widetilde{\varphi}_{-\tau(\gamma)}(\omega_{\tilde{\gamma}})\big|_{\widetilde{E}_u(\omega_{\tilde{\gamma}})\oplus\widetilde{E}_s(\omega_{\tilde{\gamma}})}$$

is the linearized Poincaré map of the periodic orbit $\widetilde{\gamma}(t)$ of the flow $\widetilde{\varphi}_t$ and $\omega_{\tilde{\gamma}} \in \mathrm{Im}(\tilde{\gamma})$ is any reference point taken in the image of $\tilde{\gamma}$.

To treat the zeta function related only to periodic rays with number of reflections $m(\gamma) \in q\mathbb{N}$, $q \geq 2$, we consider the setup introduced in [4, §4.1] and we recall it below. For $q \geqslant 2$, define the *q-reflection bundle* $\mathcal{R}_q \to M$ by

$$\mathcal{R}_q = \left(\left[S\mathbb{R}^d \setminus \left(\pi^{-1}(\mathring{D}) \cup \mathcal{D}_{\mathrm{g}}\right)\right] \times \mathbb{R}^q\right)\Big/ \approx, \tag{16}$$

where the equivalence classes of the relation $\approx$ are defined as follows. For $(x,v) \in S\mathbb{R}^d \setminus \left(\pi^{-1}(\mathring{D}) \cup \mathcal{D}_{\mathrm{g}}\right)$ and $\xi \in \mathbb{R}^q$, we set

$$[(x,v,\xi)] = \{(x,v,\xi), (x,v', A(q)\cdot\xi)\} \qquad \text{if } (x,v) \in \mathcal{D}_{\mathrm{in}},\ (x,v') \in \mathcal{D}_{\mathrm{out}},$$

where $A(q)$ is the $q \times q$ matrix with entries in $\{0,1\}$ given by

$$A(q) = \begin{pmatrix} 0 & & & 1 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}.$$

Clearly, the matrix $A(q)$ yields a shift permutation

$$A(q)(\xi_1, \xi_2, ..., \xi_q) = (\xi_q, \xi_1, ..., \xi_{q-1})$$

and

$$A(q)^q = \mathrm{Id}, \quad \mathrm{tr} A(q)^j = 0, \quad j = 1, \dots, q-1. \tag{17}$$

This indeed defines an equivalence relation since $(x, v') \in \mathcal{D}_{\text{out}}$ whenever $(x, v) \in \mathcal{D}_{\text{in}}$. Define a smooth structure of $\mathcal{R}_q$ as in [4, §4.1] and introduce the bundle

$$\mathcal{E}^q_{k,\ell} = \mathcal{E}_{k,\ell} \otimes \pi_G^* \mathcal{R}_q,$$

where $\pi_G^*\mathcal{R}_q$ is the pullback of $\mathcal{R}_q$ by π_G so $\pi_G^*\mathcal{R}_q \to G$ is a vector bundle over G. Consider a small smooth neighborhood V of K. We embed V into a smooth compact manifold without boundary N and we fix an extension of $\mathcal{R}_q$ to N. Consider any connection ∇^q on the extension of $\mathcal{R}_q$, which coincides with d^q near K, and denote by

$$P_{q,t}(z) : \mathcal{R}_q(z) \to \mathcal{R}_q(\varphi_t(z))$$

the parallel transport of ∇^q along the curve $\{\varphi_\tau(z) \ : \ 0 \leqslant \tau \leqslant t\}$. We have a smooth action of φ_t^q on $\mathcal{R}_q$, which is given by the horizontal lift of φ_t

$$\varphi_t^q(z, \xi) = (\varphi_t(z), P_{q,t}(z) \cdot \xi), \quad (z, \xi) \in \mathcal{R}_q.$$

We may lift the flow φ_t to a flow $\Phi_t^{k,\ell,q}$ on $\mathcal{E}^q_{k,\ell}$, which is defined locally near $\widetilde{K}_u$ by

$$\begin{aligned}\Phi_t^{k,\ell,q}&(\omega, u \otimes v \otimes \xi)\\ &= \Big(\widetilde{\varphi}_t(\omega),\ b_t(\omega) \cdot \Big[\big(\mathrm{d}\varphi_t(\pi_G(\omega))^{-\top}\big)^{\wedge k}(u) \otimes (\mathrm{d}\widetilde{\varphi}_t(\omega))^{\wedge \ell}(v) \otimes P_{q,t}(z) \cdot \xi\Big]\Big)\end{aligned}$$

for any $\omega = (z, E) \in G$, $u \otimes v \otimes \xi \in \mathcal{E}^q_{k,\ell}(\omega)$ and $t \in \mathbb{R}$. Following [4, §4.1], we deduce that for any periodic orbit $\gamma = (\varphi_\tau(z))_{\tau \in [0, \tau(\gamma)]}$, the trace

$$\operatorname{tr}(P_{q,\gamma}) = \operatorname{tr}(P_{q,\varphi(z)}) = \begin{cases} q & \text{if} \quad m(\gamma) = 0 \mod q, \\ 0 & \text{if} \quad m(\gamma) \neq 0 \mod q \end{cases} \tag{18}$$

is independent of z. Define the transfer operator

$$\Phi_{-t}^{k,\ell,q,*} : C^\infty(G, \mathcal{E}^q_{k,\ell}) \to C^\infty(G, \mathcal{E}^q_{k,\ell})$$

by

$$\Phi_{-t}^{k,\ell,q,*}\mathbf{u}(\omega) = \Phi_t^{k,\ell,q}[\mathbf{u}(\tilde{\varphi}_{-t}(\omega)],\ \mathbf{u} \in C^\infty(G, \mathcal{E}^q_{k,\ell})$$

and denote by $\mathbf{P}_{k,\ell,q}$ the generator of $\Phi_{-t}^{k,\ell,q,*}$. As above, we obtain the flat trace

$$\operatorname{tr}^\flat(\widetilde{\chi} e^{-t\mathbf{P}_{k,\ell,q}} \widetilde{\chi}) = q \sum_{\widetilde{\gamma}, m(\pi_G(\widetilde{\gamma})) \in q\mathbb{N}} \frac{\tau^\sharp(\gamma)\operatorname{tr}(\alpha_{\widetilde{\gamma}}^{k,\ell})\delta(t - \tau(\gamma))}{|\det(\mathrm{Id} - \widetilde{P}_\gamma)|},\ t > 0. \tag{19}$$

We close this section with the following lemma.

Lemma 2.1 (Lemma 3.1, [4]). *For any periodic orbit $\tilde{\gamma}$ of the flow $\tilde{\varphi}_t$ related to a periodic orbit γ, we have*

$$\frac{1}{|\det(\mathrm{Id} - \widetilde{P}_\gamma)|} \sum_{k=0}^{d-1} \sum_{\ell=0}^{d^2-d} (-1)^{k+\ell} \operatorname{tr}(\alpha_{\widetilde{\gamma}}^{k,\ell}) = |\det(\mathrm{Id} - P_\gamma)|^{-1/2}.$$

3. **Local trace formula.** In this section, we apply the results of [7] and [17, §6.1] for vector bundles. For simplicity, we will use the notations $\mathcal{E}_{k,\ell} = \mathcal{E}^1_{k,\ell}$, $\mathbf{P}_{k,\ell} = \mathbf{P}_{k,\ell,1}$, etc. For $\widetilde{\chi} \in C_c^\infty(\widetilde{V}_u)$ such that $\widetilde{\chi} \equiv 1$ near $\widetilde{K}_u$, by [7] and [17, §6.1], we conclude that for any integer $q \in \mathbb{N}$,

$$\widetilde{\chi}(-i\mathbf{P}_{k,\ell,q} + s)^{-1}\widetilde{\chi}$$

has a meromorphic continuation to $\mathbb{C}$. Denote by $\mathrm{Res}\,(-i\mathbf{P}_{k,\ell,q})$ the set of poles of this continuation. Then, for any constant $\beta > 0$, it was proved in [17, (6.3)] that we have the upper bound

$$\sharp \mathrm{Res}\,(-i\mathbf{P}_{k,\ell,q}) \cap \{\lambda \in \mathbb{C},\ |\mathrm{Re}\,\lambda - E| \leq 1,\ \mathrm{Im}\,\lambda \geq -\beta\} = \mathcal{O}(E^{d^2+d-1}). \tag{20}$$

In particular, there exists $C > 0$ depending on β such that

$$\sharp \mathrm{Res}\,(-i\mathbf{P}_{k,\ell,q}) \cap \{\lambda \in \mathbb{C},\ |\lambda| \leq E,\ \mathrm{Im}\,\lambda \geq -\beta\} \leq CE^{d^2+d} + C.$$

Notice that the power $d^2 + d - 1$ comes from $\dim G$. Next, for $\mathrm{Res}\,(-i\mathbf{P}_{k,\ell,q})$, we obtain as in [17] the following local trace formula.

Theorem 3.1 (Theorem 1.5 and (6.5), [17]). *For every $A > 0$ and any $q \in \mathbb{N}$, there exists a distribution $F_A^{k,\ell,q} \in \mathcal{S}'(\mathbb{R})$ supported in $[0, \infty)$ such that*

$$\sum_{\mu \in \mathrm{Res}\,(-i\mathbf{P}_{k,\ell,q}), \mathrm{Im}\,\mu > -A} e^{-i\mu t} + F_A^{k,\ell,q}(t)$$
$$= q \sum_{\widetilde{\gamma},\, m(\gamma) \in q\mathbb{N}} \frac{\tau^\sharp(\gamma)\mathrm{tr}(\alpha^{k,\ell}_{\widetilde{\gamma}})\delta(t - \tau(\gamma))}{|\det(\mathrm{Id} - \widetilde{P}_\gamma)|},\ t > 0. \tag{$21_{k,\ell,q}$}$$

Moreover, for any $\epsilon > 0$ the Fourier-Laplace transform $\hat{F}_A^{k,\ell,q}(\lambda)$ of $F_A^{k,\ell,q}(t)$ is holomorphic for $\mathrm{Im}\,\lambda < A - \epsilon$ and we have the estimate

$$|\hat{F}_A^{k,\ell,q}(\lambda)| = \mathcal{O}_{A,\epsilon,k,\ell,q}(1 + |\lambda|)^{2d^2+2d-1+\epsilon},\ \mathrm{Im}\,\lambda < A - \epsilon. \tag{22}$$

Here, $\gamma = \pi_G(\widetilde{\gamma})$.

As mentioned in [17, Section 6], the proof in [17, Section 4] with minor modifications works in the case of vector bundles. Combining the above result with Lemma 2.1, we obtain the following theorem.

Theorem 3.2. *For every $A > 0$ and any $\epsilon > 0$, there exists a distribution $F_{A,q} \in \mathcal{S}'(\mathbb{R})$ supported in $[0, \infty)$ with Fourier-Laplace transform $\hat{F}_{A,q}(\lambda)$ holomorphic for $\mathrm{Im}\,\lambda < A - \epsilon$ such that*

$$\sum_{k=0}^{d} \sum_{\ell=0}^{d^2-d} \sum_{\mu \in \mathrm{Res}\,(-i\mathbf{P}_{k,\ell,q}), \mathrm{Im}\,\mu > -A} (-1)^{k+\ell} e^{-i\mu t} + F_{A,q}(t)$$
$$= q \sum_{\gamma,\, m(\gamma) \in q\mathbb{N}} \frac{\tau^\sharp(\gamma)\delta(t - \tau(\gamma))}{|\det(\mathrm{Id} - P_\gamma)|^{1/2}},\ t > 0, \tag{23_q}$$

where $\hat{F}_{A,q}(\lambda) = \sum_{k=0}^d \sum_{\ell=0}^{d^2-d} (-1)^{k+\ell} \hat{F}_A^{k,\ell,q}(\lambda)$ satisfies estimate (22).

Choosing $q = 1$, we obtain a local trace formula for Neumann dynamical zeta function $\eta_N(s)$, introduced in Section 1. For the Dirichlet dynamical zeta function

$\eta_D(s)$ given in Section 1, we use representation (3) and applying (23_q) with $q = 1, 2$, we obtain the local trace formula

$$\sum_{k=0}^{d} \sum_{\ell=0}^{d^2-d} \sum_{\mu \in \mathrm{Res}\,(-i\mathbf{P}_{k,\ell,2}), \mathrm{Im}\,\mu > -A} (-1)^{k+\ell} e^{-i\mu t}$$

$$- \sum_{k=0}^{d} \sum_{\ell=0}^{d^2-d} \sum_{\mu \in \mathrm{Res}\,(-i\mathbf{P}_{k,\ell,1}), \mathrm{Im}\,\mu > -A} (-1)^{k+\ell} e^{-i\mu t} + F_{A,2}(t) - F_{A,1}(t)$$

$$= \sum_{\gamma} \frac{(-1)^{m(\gamma)} \tau^{\sharp}(\gamma) \delta(t - \tau(\gamma))}{|\det(\mathrm{Id} - P_\gamma)|^{1/2}}, \; t > 0. \tag{24}$$

Some resonances $\mu \in \mathrm{Res}\,(-i\mathbf{P}_{k,\ell,q}),\; k+\ell$ odd, $q = 1, 2$ may cancel with some resonances $\nu \in \mathrm{Res}\,(-i\mathbf{P}_{k,\ell,q}),\; k+\ell$ even, $q = 1, 2$, and a priori it is not clear if the meromorphic continuation of dynamical zeta functions $\eta_N(s)$ and $\eta_D(s)$ have an infinite number poles. Notice that all poles are simple and for fixed q, the cancellations in (23_q) could appear for terms with coefficients $+$ and $-$ related to $k + \ell$ even and $k + \ell$ odd, respectively. On the other hand, in (24) we have more possibilities for cancellations of poles.

4. **Strip with infinite number poles.**

Proof of Theorem 1.1. We will prove Theorem 1.1 for η_D since the argument for η_q is completely similar and simpler. After cancellation, all poles μ on the left-hand side of (24) satisfy $\mathrm{Im}\,\mu \leq \alpha = \max\{0, a_1\}$. To avoid confusion, in the following we denote by $\widetilde{\mu}$ the poles μ in (24) which are not cancelled. Assume that for some $0 < \delta < 1$ and $0 \leq k \leq q,\; 0 \leq \ell \leq q^2 - q,\; q = 1, 2$, we have estimates

$$N_{A,k,\ell,q}(r) = \sharp\{\widetilde{\mu} \in \mathrm{Res}\,(-i\mathbf{P}_{k,\ell,q}) : \; |\widetilde{\mu}| \leq r, \; -A < \mathrm{Im}\,\widetilde{\mu} \leq \alpha\} \leq P(A, k, \ell, q, \delta) r^{\delta}. \tag{25}$$

We follow the argument in [18, Section 5] and [4, Appendix B] with some modifications. Let $\rho \in C_0^\infty(\mathbb{R}, \mathbb{R}_+)$ be an even function with $\mathrm{supp}\,\rho \subset [-1, 1]$ such that

$$\rho(t) > 1 \quad \text{if} \quad |t| \; \leqslant 1/2,$$

and

$$\hat{\rho}(-\lambda) = \int \mathrm{e}^{it\lambda} \rho(t) \mathrm{d}t \geqslant 0, \quad \lambda \in \mathbb{R}.$$

Let $(\ell_j)_{j \in \mathbb{N}}$ and $(m_j)_{j \in \mathbb{N}}$ be sequences of positive numbers such that $\ell_j \geqslant d_0 = \min_{k \neq m} \mathrm{dist}\,(D_k, D_m) > 0$, $m_j \geqslant \max\{1, \frac{1}{d_0}\}$, and let $\ell_j \to \infty$, $m_j \to \infty$ as $j \to \infty$. Set

$$\rho_j(t) = \rho(m_j(t - \ell_j)), \quad t \in \mathbb{R},$$

and introduce the distribution $\mathcal{F}_{\mathrm{D}} \in \mathcal{S}'(\mathbb{R}^+)$ by

$$\mathcal{F}_{\mathrm{D}}(t) = \sum_{\gamma \in \mathcal{P}} \frac{(-1)^{m(\gamma)} \tau^{\sharp}(\gamma) \delta(t - \tau(\gamma))}{|\det(I - P_\gamma)|^{1/2}}. \tag{26}$$

The following proposition was established by Ikawa.

Proposition 4.1 (Prop. 2.3, [14]). *Suppose that the function $s \mapsto \eta_{\rm D}(s)$ cannot be prolonged as an entire function of s. Then, there exists $\alpha_0 > 0$ such that for any $\beta > \alpha_0$, we can find sequences $(\ell_j), (m_j)$ with $\ell_j \to \infty$ as $j \to \infty$ such that for all $j \geqslant 0$, one has*

$$e^{\beta \ell_j} \leqslant m_j \leqslant {\rm e}^{2\beta\ell_j} \quad \textit{and} \quad |\langle \mathcal{F}_{\rm D}, \rho_j\rangle| \geqslant {\rm e}^{-\alpha_0 \ell_j}.$$

We apply the local trace formula (24) to the function $\rho_j(t)$. For $-A \leq \operatorname{Im} \zeta \leq \alpha$, we have

$$|\hat{\rho}_j(\zeta)| = m_j^{-1} |\hat{\rho}(m_j^{-1}\zeta) e^{-i\ell_j \zeta}| \leq C_N m_j^{-1} e^{\alpha \ell_j + m_j^{-1} \max(\alpha, A)} (1 + |m_j^{-1}\zeta|)^{-N}.$$

Then, for $q = 1, 2$ and $-A \leq \operatorname{Im} \widetilde{\mu} \leq \alpha$, one obtains

$$\begin{aligned}
&\Big| \sum_{\operatorname{Im} \widetilde{\mu} > -A,\, \widetilde{\mu} \in \operatorname{Res}(-i\mathbf{P}_{k,\ell,q})} \langle e^{-i\widetilde{\mu} t}, \rho_j(t)\rangle \Big| \\
&\leqslant C_{N,A} m_j^{-1} e^{\alpha \ell_j} \int_0^\infty (1 + m_j^{-1} r)^{-N} dN_{A,k,\ell,q}(r) \\
&= -C_{N,A} m_j^{-1} e^{\alpha\ell_j} \int_0^\infty \frac{d}{dr}\Big((1 + m_j^{-1} r)^{-N}\Big) N_{A,k,l,q}(r) dr \\
&\leqslant B_{N,A} P(A, k, \ell, q, \delta) m_j^{-(1-\delta)} e^{\alpha \ell_j} \int_0^\infty (1 + y)^{-N-1} y^\delta dy \\
&= A_N P(A, k, \ell, q, \delta) m_j^{-(1-\delta)} e^{\alpha\ell_j} \leq D_{A,k,\ell,q,\delta} e^{(-\beta(1-\delta) + \alpha)\ell_j}.
\end{aligned}$$

Next, applying (22), we have

$$\langle F_{A,q}, \rho_j\rangle = \int_{\mathbb{R}} \hat{F}_{A,q}(-\zeta) \hat{\rho}_j(\zeta) d\zeta = \int_{\mathbb{R} + i(\epsilon - A)} \hat{F}_{A,q}(-\zeta)\hat{\rho}_j(\zeta) d\zeta$$

and choosing $M = 2d^2 + 2d + 1$, we deduce

$$\begin{aligned}
|\langle F_{A,q}, \rho_j\rangle| &\leq C_{M,A,q} m_j^{-1} e^{(\epsilon - A)\ell_j} e^{m_j^{-1} \max\{A - \epsilon, \alpha\}} \\
&\quad \times \int (1 + |\zeta|)^{2d^2 + 2d - 1 + \epsilon} (1 + |m_j^{-1}\zeta|)^{-M} d\zeta \\
&\leq D_{M,A,q} e^{(\epsilon - A)\ell_j} m_j^{2d^2 + 2d - 1 + \epsilon} \leq D_{M,A,q} e^{(\epsilon - A)\ell_j} e^{2(2d^2 + 2d - 1 + \epsilon)\beta \ell_j}.
\end{aligned}$$

If the function η_D cannot be prolonged as an entire function, we may apply Proposition 4.1. Taking together the above estimates and summing for $0 \leq k \leq d$, $0 \leq \ell \leq d^2 - d$ and $q = 1, 2$, we get

$$D_A e^{(-\beta(1-\delta) + \alpha)\ell_j} + E_A e^{(\epsilon - A)\ell_j} e^{2(2d^2 + 2d - 1 + \epsilon)\beta\ell_j} \geq e^{-\alpha_0 \ell_j}.$$

Here, the constants D_A and E_A depend on A, but they are independent of ℓ_j. Choose $\beta > \frac{\alpha_0 + \alpha}{1 - \delta}$ and fix β and $0 < \epsilon < 1$. Next, choose

$$A > 2(2d^2 + 2d - 1 + \epsilon)\beta + \epsilon + \alpha_0.$$

Fixing A, for $\ell_j \to \infty$ we obtain a contradiction. This completes the proof of Theorem 1.1 for η_D.

To deal with $\operatorname{Res} \eta_q$, $q \geq 2$, we choose a periodic ray γ_0 with q reflections, $\ell_j = j\tau^\sharp(\gamma_0)$ and $m_j = e^{\beta \ell_j}$. The existence of a periodic ray with q reflexions follows from the fact that for every configuration $\xi \in \Sigma_A$, there exists a unique ray

$\gamma(\xi)$ with successive reflexion points on $....\partial D_{j-1}, \partial D_j, \partial D_{j+1}, ...$ (see the appendix for the definition of Σ_A and [12]). We apply the lower bound

$$\Big|\langle \sum_{\gamma,\, m(\gamma)\in q\mathbb{N}} \frac{\tau^\sharp(\gamma)\delta(t-\tau(\gamma))}{|\det(\mathrm{Id}-P_\gamma)|^{1/2}}, \rho_j\rangle\Big| \geq c e^{-c_0 \ell_j},\ \forall j \geq 0$$

with $c > 0, c_0 > 0$ independent of ℓ_j. For $q = 1$, we choose $\ell_j = j\tau^\sharp(\gamma)$, $m_j = e^{\beta \ell_j}$ with some periodic ray γ and obtain the above estimate. Repeating the argument for η_D, we prove (4). □

Proof of Theorem 1.3. We follow the approach of F. Naud in [18, Appendix A]. Let $0 \leq \rho \in C_0^\infty(-1, 1)$ be the function introduced above. For $\xi \in \mathbb{R}$ and $t > \max\{d_0, 1\}$, introduce the function

$$\psi_{t,\xi}(s) = e^{i\xi s}\rho(s-t),\ \xi \in \mathbb{R}.$$

In the proof we fix $q = 1$. We apply the trace formula (23_q) to $\psi_{t,\xi}$. As above, denote by $\widetilde{\mu}$ the poles which are not cancelled on the left-hand side of (23_q). Assume that for $0 \leq k \leq d$ and $0 \leq \ell \leq d^2 - d$, we have

$$\sharp\{\widetilde{\mu} \in \mathrm{Res}\,(-i\mathbf{P}_{k,l}) :\ -A-\epsilon \leq \mathrm{Im}\,\widetilde{\mu} \leq \alpha\} = P(A, k, \ell, \epsilon) < \infty. \tag{27}$$

First, we have

$$|\hat{\psi}_{t,\xi}(\zeta)| \leq C_N e^{t\mathrm{Im}\,\zeta + |\mathrm{Im}\,\zeta|}(1 + |\mathrm{Re}\,\zeta - \xi|)^{-N}.$$

For $-A \leq \mathrm{Im}\,\widetilde{\mu} \leq \alpha$ and $N = 1$, the sum of terms involving poles $\widetilde{\mu}$ in (23_q) can be bounded by $\frac{C_1 e^{\alpha t}}{1+|\xi|}$ with constant $C_1 > 0$ depending on $P(A, k, \ell, \epsilon)$ and $\exp(\max\{A, \alpha\})$. Second, by using (22) for $\hat{F}_{A,1}$, one deduces

$$|\langle F_{A,1}, \psi_{t,\xi}\rangle| \leq C_2 e^{(\epsilon - A)(t-1)}(1 + |\xi|)^{2d^2+2d-1+\epsilon}.$$

Setting

$$S(t, \xi) = \sum_\gamma \frac{e^{i\xi\tau(\gamma)}\tau^\sharp(\gamma)\rho(\tau(\gamma)-t)}{|\det(\mathrm{Id}-P_\gamma)|^{1/2}},$$

we get

$$|S(t,\xi)| \leq \frac{C_1 e^{\alpha t}}{1+|\xi|} + C_A e^{-(A-\epsilon)t}(1+|\xi|)^{2d^2+2d-1+\epsilon}.$$

Now consider the Gaussian weight

$$G(t,\sigma) = \sigma^{1/2}\int_{\mathbb{R}} |S(t,\xi)|^2 e^{-\sigma\xi^2/2} d\xi,\ 0 < \sigma < 1.$$

The estimate for $|S(t, \xi)|$ yields

$$|S(t,\xi)|^2 \leq \frac{2C_1^2 e^{2\alpha t}}{(1+|\xi|)^2} + 2C_A^2 e^{-2(A-\epsilon)t}(1+|\xi|)^{2(2d^2+2d-1+\epsilon)}$$

and

$$G(t,\sigma) \leq C_1' \sigma^{1/2} e^{2\alpha t} + C_A' \sigma^{-(2d^2+2d-1+\epsilon)} e^{-2(A-\epsilon)t}. \tag{28}$$

On the other hand, taking into account only the terms with $\tau(\gamma) = \tau(\gamma')$, we get

$$\begin{aligned} G(t,\sigma) =& \sqrt{2\pi}\sum_\gamma\sum_{\gamma'} \frac{\tau^\sharp(\gamma)\tau^\sharp(\gamma')e^{-(\tau(\gamma)-\tau(\gamma'))^2/2\sigma}\rho(\tau(\gamma)-t)\rho(\tau(\gamma')-t)}{|\det(\mathrm{Id}-P_\gamma)|^{1/2}|\det(\mathrm{Id}-P_{\gamma'})|^{1/2}} \\ \geq& c \sum_{t-1/2\leq\tau(\gamma)\leq t+1/2} \tau^\sharp(\gamma)|\det(\mathrm{Id}-P_\gamma)|^{-1} \end{aligned} \tag{29}$$

with $c > 0$ independent of t and σ.

Set $\tau(\gamma) = T_\gamma$, $\tau^\sharp(\gamma) = T^\sharp_\gamma$, $a_\gamma = \frac{T^\sharp_\gamma}{|\det(\mathrm{Id} - P_\gamma)|}$. Recall that b_1 is the abscissa of convergence of Dirichlet series (5) with $q = 1$.

We need the following lemma.

Lemma 4.2. *Let $\epsilon > 0$ be sufficiently small and $b_1 \neq 0$. Then there exists a sequence $t_j \to \infty$ such that*

$$\sum_{t_j - 1/2 \leq T_\gamma \leq t_j + 1/2} a_\gamma \geq e^{(b_1 - 2\epsilon)t_j}. \tag{30}$$

On the other hand, for $b_1 = 0$ and small $u > 0$, there exists a sequence $t_j \to \infty$ such that we have estimate (30) with b_1 replaced by $-u/2$

Proof. We consider three cases.

Case 1. $b_1 < 0$.

Let the lengths of the periodic rays be arranged as follows:

$$T_1 \leq T_2 \leq \ldots \leq T_n \leq \ldots.$$

It is well known (see for instance [5]) that

$$b_1 = \limsup_{n \to \infty} \frac{\log |\sum_{T_n \leq T_\gamma} a_\gamma|}{T_n}.$$

We fix a small $\epsilon > 0$ so that $-\delta = b_1 - 3\epsilon/2 < 0$. There exists an increasing sequence $n_1 < n_2 < \ldots < n_m < \ldots$ such that $\lim n_j = +\infty$ and

$$\frac{\log |\sum_{T_{n_j} \leq T_\gamma} a_\gamma|}{T_{n_j}} \geq b_1 - \epsilon. \tag{31}$$

Choose n_1 large so that

$$1 > e^{-\delta} + 2e^{-\frac{\epsilon}{2}T_{n_j}},\ e^{\frac{\epsilon}{2}T_{n_j}} \geq e^{\delta/2} \text{ for } j \geq 1.$$

Set $q_1 = T_{n_1}$, and write

$$\sum_{q_1 \leq T_\gamma} a_\gamma = \sum_{k=0}^{\infty} \sum_{q_1 + k \leq T_\gamma < q_1 + k + 1} a_\gamma.$$

Assume that we have the estimates

$$\sum_{q_1 + k \leq T_\gamma < q_1 + k + 1} a_\gamma \leq e^{-\delta(q_1 + k)},\ \forall k \geq 0. \tag{32}$$

Then,

$$\sum_{q_1 \leq T_\gamma} a_\gamma \leq e^{-\delta q_1} \sum_{k=0}^{\infty} e^{-k\delta} = e^{-\delta q_1} \frac{1}{1 - e^{-\delta}} < \frac{1}{2} e^{(-\delta + \epsilon/2)q_1}.$$

Since $-\delta + \epsilon/2 = b_1 - \epsilon$, we obtain a contradiction with (31) for T_{n_1}. Consequently, there exists at least one $k_1 \geq 0$ such that

$$\sum_{q_1 + k_1 \leq T_\gamma < q_1 + k_1 + 1} a_\gamma > e^{-\delta(q_1 + k_1)}. \tag{33}$$

The series $\sum_{T_\gamma \geq q_1 + k_1 + 2} a_\gamma e^{-\lambda T_\gamma}$ has the same abscissa of convergence b_1. We repeat the procedure, choosing $q_2 \geq q_1 + k_1 + 2$, and obtain the existence of $k_2 \geq 0$ such that

$$\sum_{q_2 + k_2 \leq T_\gamma < q_2 + k_2 + 1} a_\gamma > e^{-\delta(q_2 + k_2)}.$$

By iteration, we find two sequences $\{q_j\}$, $\{k_j\}$ such that

$$q_{j+1} \geq q_j + k_j + 2,$$

and a sequence of disjoint intervals

$$[q_j + k_j, q_j + k_j + 1],\ j = 1, 2, \ldots$$

so that

$$\sum_{q_j + k_j \leq T_\gamma \leq q_j + k_j + 1} a_\gamma > e^{-\delta(q_j + k_j)}. \tag{34}$$

The periods q_j may change in the above procedure, but for simplicity we use the same notation. Choosing $t_j = q_j + k_j + 1/2$, we deduce (30).

Case 2. $b_1 > 0$.

For b_1, we have the representation

$$b_1 = \limsup_{n \to \infty} \frac{\log |\sum_{T_\gamma \leq T_n} a_\gamma|}{T_n}.$$

We fix a small $\epsilon > 0$ so that $d_1 = b_1 - 3\epsilon/2 > 0$. There exists an increasing sequence $n_1 < n_2 < \ldots < n_m < \ldots$ such that $\lim n_j = +\infty$, $\frac{e^{d_1}}{e^{d_1} - 1} < e^{\frac{\epsilon}{2}[T_{n_1}]}$, and

$$\frac{\log |\sum_{T_\gamma \leq T_{n_j}} a_\gamma|}{T_{n_j}} \geq b_1 - \epsilon. \tag{35}$$

We get

$$\sum_{T_\gamma \leq T_{n_1}} a_\gamma \leq \sum_{k=0}^{[T_{n_1}]} \sum_{k < T_\gamma \leq k+1} a_\gamma.$$

Assume that for $k = 0, \ldots, [T_{n_1}]$ we have

$$\sum_{k < T_\gamma \leq k+1} a_\gamma \leq e^{d_1 k}.$$

This implies

$$\sum_{T_\gamma \leq T_{n_1}} a_\gamma \leq \sum_{k=0}^{[T_{n_1}]} e^{d_1 k} = \frac{e^{([T_{n_1}]+1)d_1} - 1}{e^{d_1} - 1} < \frac{e^{d_1} e^{-\frac{\epsilon}{2}[T_{n_1}]}}{e^{d_1} - 1} e^{(b_1 - \epsilon)T_{n_1}}$$

and we obtain a contradiction with (35) and T_{n_1}. Thus, for some $0 \leq k_1 \leq [T_{n_1}]$ we have

$$\sum_{k_1 < T_\gamma \leq k_1 + 1} a_\gamma \geq e^{(b_1 - 3\epsilon/2)k_1}.$$

Following this procedure, we construct a sequence of integers $\{k_j\}$, $k_{j+1} \geq T_{k_j} + 1$ satisfying

$$\sum_{k_j \leq T_\gamma \leq k_j + 1} a_\gamma \geq e^{(b_1 - 3\epsilon/2)k_j}$$

and choosing $t_j = k_j + 1/2$, we arrange (30) for large k_j.

Case 3. $b_1 = 0$.

For small $u > 0$, consider the Dirichlet series

$$\eta_u(s) = \sum_\gamma \frac{T_\gamma^\sharp e^{-(s+u)T_\gamma}}{|\det(\mathrm{Id} - P_\gamma)|} = \sum_\gamma a_\gamma e^{-uT_\gamma} e^{-sT_\gamma}.$$

This series has abscissa de convergence $-u < 0$ and we may apply the results of Case 1. For a suitable sequence $t_j \to \infty$ depending on $-u$, we obtain the estimates

$$e^{-u(t_j-1/2)} \sum_{t_j-1/2 \leq T_\gamma \leq t_j+1/2} a_\gamma \geq \sum_{t_j-1/2 \leq T_\gamma \leq t_j+1/2} a_\gamma e^{-uT_\gamma} \geq e^{(-u-2\epsilon)t_j}.$$

Consequently,

$$\sum_{t_j-1/2 \leq T_\gamma \leq t_j+1/2} a_\gamma \geq e^{-u/2} e^{-2\epsilon t_j} > e^{(-u/2-2\epsilon)t_j}.$$

□

Going back to the proof of Theorem 1.3, assume first that $b_1 \neq 0$. Therefore, from (28) and (29) with $t = t_j$, we obtain

$$c_1 \sigma^{1/2} e^{2\alpha t_j} + c_2 \sigma^{-(2d^2+2d-1+\epsilon)} e^{-2(A-\epsilon)t_j} \geq e^{(b_1-2\epsilon)t_j} \tag{36}$$

with constants $c_1, c_2 > 0$ independent of t_j. Now choose

$$\sigma = c_1^{-2} e^{2(b_1-3\epsilon-2\alpha)t_j} < 1.$$

Since

$$b_1 - 3\epsilon - 2\alpha - (b_1 - 2\epsilon) + 2\epsilon \leq \epsilon,$$

we have

$$e^{-\epsilon t_j} + c_3 e^{-2(2d^2+2d-1/2+\epsilon)(b_1-3\epsilon-2\alpha)t_j} e^{-2(A-(1/2)\epsilon)t_j} \geq 1.$$

Taking

$$A = -(2d^2+2d-1/2)(b_1-2\alpha) + 3\epsilon\Big(2d^2+2d - \frac{b_1-2\alpha}{3} + \epsilon\Big)$$

and letting $t_j \to +\infty$, we obtain a contradiction. Consequently, for some $0 \leq k_0 \leq 0$, $0 \leq \ell_0 \leq d^2 - d$, setting $\tilde{\epsilon} = 3\epsilon(2d^2+2d-\frac{b_1-2\alpha}{3}+\epsilon)+\epsilon$, we have

$$\sharp\{\widetilde{\mu} \in \text{Res}\,(-i\mathbf{P}_{k_0,l_0}) :\ \text{Im}\,\widetilde{\mu} > (2d^2+2d-1/2)(b_1-2\alpha) - \tilde{\epsilon}\} = \infty.$$

This implies (6) with ϵ replaced by $\tilde{\epsilon}$, observing that the poles $\widetilde{\mu} \in \text{Res}\,(-i\mathbf{P}_{k_0,\ell_0})$ coincide with the poles $\widetilde{\lambda}$ of the meromorphic continuation of $\eta_1(-i\lambda)$.

For $b_1 = 0$, the estimates (30) hold with b_1 replaced by $-u/2$. The argument in the Case 1 implies

$$\sharp\{\mu_j \in \text{Res}\ \eta_1 :\ \text{Re}\,\mu_j > (2d^2+2d-1/2)(-2\alpha) - (\epsilon + (2d^2+2d-1/2)u/2)\} = \infty.$$

For small u, we arrange $(2d^2+2d-1/2)u/2 < \epsilon$, and since ϵ is arbitrary, we obtain (6) with $b_1 = 0$. This completes the proof of Theorem 1.3. □

Appendix. Here, we prove Proposition 1.4.

Proof of Proposition 1.4. First,

$$\det(\text{Id} - P_\gamma) = \det(\text{Id} - D_x\varphi_{T_\gamma}|_{E_s(x)})\det(\text{Id} - D_x\varphi_{T_\gamma}|_{E_u(x)})$$
$$= \det(D_x\varphi_{T_\gamma}|_{E_u(x)})\det(\text{Id} - D_x\varphi_{T_\gamma}|_{E_s(x)})\det(D_x\varphi_{-T_\gamma}|_{E_u(x)} - \text{Id}),\ x \in \gamma.$$

Consequently,

$$|\det(\text{Id} - P_\gamma)|^{-1} = |\det D_x\varphi_{T_\gamma}|_{E_u(x)}|^{-1}$$
$$\times |\det(\text{Id} - D_x\varphi_{T_\gamma}|_{E_s(x)})|^{-1} |\det(\text{Id} - D_x\varphi_{-T_\gamma}|_{E_u(x)})|^{-1}.$$

For large T_γ we have

$$\|D_x\varphi_{T_\gamma}|_{E_s(x)}\| \leq Ce^{-\delta T_\gamma},\ \|D_x\varphi_{-T_\gamma}|_{E_u(x)}\| \leq Ce^{-\delta T_\gamma},\ \delta > 0, \forall T_\gamma$$

with constants $C > 0$, $\delta > 0$ independent of T_γ since the flow φ_t is uniformly hyperbolic (see [4, Appendix A]). Thus, for large T_γ we obtain

$$c_1|\det D_x\varphi_{T_\gamma}|_{E_u(x)}|^{-1} \leq |\det(\mathrm{Id} - P_\gamma)|^{-1} \leq C_1|\det D_x\varphi_{T_\gamma}|_{E_u(x)}|^{-1} \tag{A.1}$$

with $0 < c_1 < C_1$ independent of T_γ. We have

$$\det D_x\varphi_{T_\gamma}|_{E_u(x)} = e^{d_\gamma},\ x \in \gamma$$

with

$$d_\gamma = \log\Big(\lambda_{1,\gamma}...\lambda_{d-1,\gamma}) > 0,$$

where $\lambda_{j,\gamma}$ are the eigenvalues of $D_x\varphi_{T_\gamma}|_{E_u(x)}$ with modulus greater than 1. The above estimate shows that the abscissa of convergence of the series

$$\sum_\gamma T_\gamma^\sharp e^{-sT_\gamma+\delta_\gamma},\ \delta_\gamma = -d_\gamma,\ \mathrm{Re}\, s \gg 1$$

coincides with b_1.

Our purpose is to express b_1 by some dynamical characteristics related to symbolic dynamics for several disjoint strictly convex obstacles. To do this, we recall some well known results and we refer to [13], [12], [15] and [20] for more details. Let $A(i,j)_{i,j=1,\dots,r}$ be an $r \times r$ matrix such that

$$A(i,j) = \begin{cases} 1 \text{ if } i \neq j, \\ 0 \text{ if } i = j. \end{cases}$$

Introduce the spaces

$$\Sigma_A = \{\xi = \{\xi_i\}_{i=-\infty}^{\infty},\ \xi_i \in \{1,,...,r\},\ A(\xi_i, \xi_{i+1}) = 1,\ \forall i \in \mathbb{Z}\},$$
$$\Sigma_A^+ = \{\xi = \{\xi_i\}_{i=0}^{\infty},\ \xi_i \in \{1,,...,r\},\ A(\xi_i, \xi_{j+1}) = 1,\ \forall i \geq 0\}.$$

Given $0 < \theta < 1$, define a metric d_θ on Σ_A by $d_\theta(\xi,\eta) = 0$ if $\xi = \eta$ and $d_\theta(\xi,\eta) = \theta^k$ if $\xi \neq \eta$, and let k be the maximal integer such that $\xi_i = \eta_i$ for $|i| < k$. Similarly, we define a metric d_θ^+ on Σ_A^+. Following [20, Chapter 1], for a function $F : \Sigma_A \to \mathbb{C}$, define

$$\mathrm{var}_k F = \sup\{|F(\xi) - F(\eta)| :\ \xi_i = \eta_i,\ |i| < k\}$$

and for $G : \Sigma_A^+ \to \mathbb{C}$, define

$$\mathrm{var}_k G = \sup\{|G(\xi) - G(\eta)| :\ \xi_i = \eta_i,\ 0 \leq i < k\}.$$

Let $F_\theta(\Sigma_A)$, $F_\theta(\Sigma_A^+)$ be the set of Lipschitz functions with respect to metrics d_θ, d_θ^+, respectively, with norm

$$|||f|||_\theta = \|f\|_\infty + \|f\|_\theta,\ \|f\|_\theta = \sup_{k\geq 0} \frac{\mathrm{var}_k f}{\theta^k}.$$

Let σ_A be a shift on Σ_A and Σ_A^+ given by

$$(\sigma_A\xi)_i = \xi_{i+1},\ \forall i \in \mathbb{Z},\ (\sigma_A\xi)_i = \xi_{i+1},\ \forall i \geq 0,$$

respectively. For every $\xi \in \Sigma_A$, there exists a unique reflecting ray $\gamma(\xi)$ with successive reflection points on $....\partial D_{j-1}, \partial D_j, \partial D_{j+1}, ...,$ where the order of reflections is determined by the sequence ξ (see [12]). If $(P_j(\xi))_{j=-\infty}^{\infty}$ are the reflexion points of $\gamma(\xi)$, we define the function

$$f(\xi) = \|P_0(\xi) - P_1(\xi)\|.$$

It was proved in [12, Section 3] and [23, Section 3] that one can construct a sequence of phase functions $\{\varphi_{\xi,j}(x)\}_{j=-\infty}^{\infty}$ such that for each j, the phase $\varphi_{\xi,j}$ is smooth in a neighborhood $\mathcal{U}_{\xi,j}$ of the segment $[P_j(\xi), P_{j+1}(\xi)]$ in $\mathbb{R}^d \setminus \mathring{D}$ and

(i) $\|\nabla \varphi_{\xi,j}(x)\| = 1$ on $\mathcal{U}_{\xi,j}$,

(ii) $\nabla \varphi_{\xi,j}(P_j(\xi)) = \frac{P_{j+1}(\xi)-P_j(\xi)}{\|P_{j+1}(\xi)-P_j(\xi)\|}$,

(iii) $\varphi_{\xi,j} = \varphi_{\xi,j+1}$ on $\partial D_{j+1} \cap \mathcal{U}_{\xi,j} \cap \mathcal{U}_{\xi,j+1}$,

(iv) for each $x \in \mathcal{U}_{\xi,j}$, the surface $C_{\xi,j}(x) = \{y \in \mathcal{U}_{\xi,j} : \varphi_{\xi,j}(x) = \varphi_{\xi,j}(y)\}$ is strictly convex with respect to its normal fields $\nabla \varphi_{\xi,j}$.

Denote by $\kappa_j(\xi)$, $j = 1, ..., d-1$, the principal curvatures at $P_0(\xi)$ of $C_{\xi,0}(x)$, and introduce

$$g(\xi) = -\log \prod_{j=1}^{d-1} (1 + f(\xi)\kappa_j(\xi)).$$

Then,

$$\prod_{j=1}^{d-1} \lambda_{j,\gamma(\xi)} = \prod_{k=1}^{m(\gamma(\xi))} \prod_{j=1}^{d-1} (1 + f(\sigma_A^k \xi)\kappa_j(\sigma_A^k \xi)).$$

It follows form the exponential instability of the billiard ball map (see [12], [26]) that $f(\xi), g(\xi)$ become functions in $F_\theta(\Sigma_A)$ with $0 < \theta < 1$ depending on the geometry of D. We define

$$S_n h(\xi) = h(\xi) + h(\sigma_A \xi) + ... + h(\sigma_A^{n-1}\xi)$$

and for a periodic ray $\gamma(\xi)$, one obtains

$$T_{\gamma(\xi)} = S_{m(\gamma(\xi))} f(\xi),\ \delta_{\gamma(\xi)} = S_{m(\gamma(\xi))} g(\xi).$$

Consider the zeta function

$$Z(s) = \Big(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{\sigma_A^n \xi = \xi} e^{S_n(-sf(\xi)+g(\xi))}\Big),\ \mathrm{Re}\, s \gg 1$$

and observe that

$$-\frac{d}{ds} Z(s) = \sum_\gamma T_\gamma^\sharp e^{-sT_\gamma + \delta_\gamma}.$$

Next, it is well known (see for instance [20, Chapter 1]) that given $h \in F_\theta(\Sigma_A)$, there exist functions $\tilde{h}, \chi \in F_{\theta^{1/2}}(\Sigma_A)$ such that

$$h(\xi) = \tilde{h}(\xi) + \chi(\sigma_A \xi) - \chi(\xi)$$

and $\tilde{h}(\xi) \in F_{\theta^{1/2}}(\Sigma_A^+)$ depends only on the coordinates $(\xi_0, \xi_1, ...)$. We denote this property by $h \sim \tilde{h}$. Choose $\tilde{f} \sim f$, $\tilde{g} \sim g$ with $\tilde{f}, \tilde{g} \in F_{\theta^{1/2}}(\Sigma_A^+)$ and write

$$Z(s) = \Big(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{(\sigma_A^+)^n \xi = \xi} e^{S_n(-s\tilde{f}(\xi)+\tilde{g}(\xi))}\Big).$$

The pressure $\mathrm{P}(F)$ of a function $F \in C(\Sigma_A)$ is defined by

$$\mathrm{P}(F) = \sup_\nu \Big(h(\nu, \sigma_A) + \int_{\Sigma_A} F d\nu\Big),$$

where $h(\nu, \sigma_A)$ is the measure entropy of σ_A with respect to ν, and the supremum is taken over all probability measures ν on Σ_A invariant with respect to σ_A.

Following [20, Chapter 6], consider the suspended flow $\sigma_t^f(\xi, s) = (\xi, s + t)$ on the space

$$\Sigma_A^f = \{(\xi, s) : \xi \in \Sigma_A,\ 0 \le s \le f(\xi)\}$$

with identification $(\xi, f(\xi)) \sim (\sigma_A(\xi), 0)$. For a function $G \in C(\Sigma_A^f)$, define the pressure

$$\mathrm{P}(G) = \sup_{\nu_f}\{h(\nu_f, \sigma_t^f) + \int_{\Sigma_A^f} G d\nu_f\}, \tag{A.2}$$

where $h(\nu_f, \sigma_t^f)$ is the measure entropy, and the supremum is taken over all probability measures ν_f on Σ_A^f invariant with respect to σ_t^f. The suspended flow σ_t^f is weakly mixing if there are no $t \in \mathbb{R} \setminus \{0\}$ with the property

$$\frac{t}{2\pi} f(\xi) \sim M(\xi),$$

where $M(\xi) \in C(\Sigma_A : \mathbb{Z})$ has only integer values. According to [26, Lemma 5.2] and [21, Lemma 1], the flow σ_t^f is *weakly mixing*.

Applying the results of [20, Chapter 6], we deduce that the abscissa of convergence b_1 of $Z(s)$ is determined as the root of the equation

$$\mathrm{P}(-s\tilde{f} + \tilde{g}) = \mathrm{P}(-sf + g) = 0$$

with respect to s. This root is unique since $s \to \mathrm{P}(-sf + g)$ is decreasing. Introduce the function

$$G(\xi, y) = -\frac{1}{2}\sum_{j=1}^{d-1} \frac{\kappa_j(\xi)}{1 + \kappa_j(\xi) y}. \tag{A.3}$$

Clearly,

$$g(\xi) = 2\int_0^{f(\xi)} G(\xi, y) dy.$$

Then, [20, Proposition 6.1] says that $\mathrm{P}(-b_1 f + g) = 0$ is equivalent to $b_1 = \mathrm{P}(2G)$. With the same argument, we show that $a_1 = \mathrm{P}(G)$. This completes the proof of Proposition 1.4. □

It is easy to find a relation between b_1 and $\mathrm{P}(g)$. Repeating the argument of [23, Section 3], one obtains that there exist probability measures ν_g, ν_0 on Σ_A invariant with respect to σ_A such that

$$\frac{\mathrm{P}(g)}{\int f(\xi) d\nu_g} \le b_1 \le \frac{\mathrm{P}(g)}{\int f(\xi) d\nu_0}.$$

Consequently, b_1 has the same sign as $\mathrm{P}(g)$.

We close this appendix by proving that $b_1 = b_2$. Consider the zeta function

$$Z_1(s) = \Big(\sum_{n=1}^{\infty} \frac{1}{n} \sum_{(\sigma_A^+)^n \xi = \xi} e^{S_n(-s\tilde{f}(\xi) + \tilde{g}(\xi) + i\pi)}\Big)$$

related to (9). Introduce the complex Ruelle operator

$$(\mathcal{L}_s u)(\xi) = \sum_{\sigma_A^+ \eta = \xi} e^{(-s\tilde{f} + \tilde{g} + i\pi)(\eta)} u(\eta),\ u \in F_{\theta^{1/2}}(\Sigma_A^+).$$

Then, for $s = b_1$, this operator has no eigenvalues 1 since this implies that the operator

$$(L_{b_1}u)(\xi) = \sum_{\sigma_A^+\eta=\xi} e^{(-b_1\tilde{f}+\tilde{g})(\eta)}u(\eta)$$

will have eigenvalue (-1). This is impossible because from $\mathrm{P}(-b_1\tilde{f}+\tilde{g}) = 0$, one deduces that L_{b_1} has eigenvalue 1, and all other eigenvalues of L_{b_1} have modulus strictly less than 1 (see [20, Theorem 2.2]). This shows that the function $Z_1(s)$ is analytic for $s = b_1$, hence (9) has the same property. Finally, similarly to (3), we write the function (9) as a difference of two Dirichlet series with abscissas of convergence b_1 and b_2. Therefore, the inequality $b_2 < b_1$ leads to a contradiction.

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